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Parametric Excitation of Subharmonic Oscillations

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Subharmonic oscillations of order one-half for a single-degree-of-freedom system with quadratic, cubic, and quartic nonlinearities under parametric excitation are investigated. Two approximate methods (multiple scales and generalized synchronization) are used for comparison. The modulation equations (reduced equations) of the amplitudes and the phases are obtained. Steady-state solutions (periodic solutions) and their stability are determined. Numerical solutions are carried out, and graphical representations of the results are presented and discussed. The results obtained by the two methods are in excellent agreement.

1. INTRODUCTION

Excitations produced by time-dependent parameters in the governing equations are called parametric excitations. In contrast with the case of external excitations, which lead to inhomogeneous differential equations with constant or slowly varying coefficients, parametric excitations lead to homogeneous differential equations with rapidly varying coefficients, usually periodic ones.

The problem of parametric excitation arises in many branches of physics and engineering. For a comprehensive review of the response of single- and multi-degree-of-freedom systems to parametric excitations, see Evan-Iwanowski (1976), Nayfeh and Mook (1979), Ibrahim (1985), Schmidt and Tondl (1986), Zavodney (1987), Balbi (1973), Haag (1962), Elnaggar and Hamd-Allah (1982), Elnaggar (1985), Elnaggar and El-Basyouny (1992, 1993, 1995), and Elnaggar and El-Diriny (1995).

Zavodney and Nayfeh (1988), using the method of multiple scales, studied the response of a single-degree-of-freedom system with quadratic and cubic nonlinearities to a fundamental parametric resonance. Zavodney

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et al. (1989), using the same method, studied the case of a principal parametric excitation.

In this paper, we investigate subharmonic oscillations of order one-half (principal parametric resonance) for a single-degree-of-freedom system with quadratic, cubic, and quartic nonlinearities under a parametric excitation. The quadratic term may be due to curvature or an asymmetric material nonlinearity, whereas the cubic and quartic terms may be due to mid-plane stretching or a symmetric material nonlinearity. The parametric term may be due to a harmonic axial load. Two approximate methods are used to find two first-order ordinary differential equations describing the modulation of the amplitudes and the phases. Steady-state solutions (periodic solutions) and their stability are determined. Numerical calculations are carried out. The results obtained by the two methods are in excellent agreement.

2. FORMULATION OF THE PROBLEM

Subharmonic oscillations (periodic oscillations) of order one-half for a single-degree-of-freedom system with viscous damping and quadratic, cubic and quartic nonlinearities under a parametric excitation can be modeled by a second-order ordinary differential equation of the form

$$\ddot{u} + \omega_0^2 u + \epsilon(2\mu\dot{u} + \alpha_1 u^2 + \alpha_2 u^3 + \alpha_3 u^4 + \alpha_4 u \cos \Omega t) = 0 \qquad (1)$$

where the dots indicate differentiation with respect to time t, ϵ is a small parameter, ω_0 is the linear natural frequency, μ is the coefficient of viscous damping, α_1, α_2, and α_3 are the coefficients of the nonlinear terms, and α_4 and Ω are the amplitude and frequency of the parametric excitation, respectively.

3. THE METHOD OF MULTIPLE SCALES

A first-order uniform solution of equation (1) is sought by using the method of multiple scales (Nayfeh and Mook, 1979) in the form

$$u(t; \epsilon) = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \cdots \qquad (2)$$

where $T_0 = t$ is a fast scale associated with changes occurring at the frequencies ω_0 and Ω, and $T_1 = \epsilon t$ is a slow scale associated with modulations in the amplitude and the phase caused by the nonlinearity, damping, and oscillation. In terms of the T_1, the time derivatives become

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \cdots, \qquad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \cdots \qquad (3)$$

where $D_n = \partial/\partial T_n$. Substituting equations (2) and (3) into equation (1) and equating coefficients of like powers of ϵ, one obtains

$$D_0^2u_0 + \omega_0^2u_0 = 0 \tag{4}$$

$$D_0^2u_1 + \omega_0^2u_1 = -2D_0D_1u_0 - 2\mu D_0u_0 - \alpha_1u_0^2 - \alpha_2u_0^3 - \alpha_3u_0^4 - \alpha_4u_0 \cos \Omega T_0 \tag{5}$$

The solution of equation (4) can be expressed in the complex form

$$u_0 = A(T_1)e^{i\omega_0T_0} + \bar{A}(T_1)e^{-i\omega_0T_0} \tag{6}$$

where $\bar{A}$ is the complex conjugate of A. Then, equation (5) becomes

$$\begin{aligned} D_0^2u_1 + \omega_0^2u_1 = &-[2i\omega_0(D_1A + \mu A) + 3\alpha_2A^2\bar{A}]e^{i\omega_0T_0} \\ &- (\alpha_1 + 4\alpha_3 A\bar{A})A^2e^{2i\omega_0T_0} \\ &- \alpha_2A^3e^{3i\omega_0T_0} - \alpha_3A^4e^{4i\omega_0T_0} - \frac{1}{2}\alpha_4Ae^{i(\Omega+\omega_0)T_0} \\ &- \frac{1}{2}\alpha_4\bar{A}e^{i(\Omega-\omega_0)T_0} - 2A\bar{A}(\alpha_1 + 3\alpha_3A\bar{A}) + \text{c.c.} \end{aligned} \tag{7}$$

where c.c. stands for the complex conjugate of the preceding terms. Any particular solution of equation (7) contains secular terms and small-divisor terms when $\Omega \approx 2\omega_0$. To treat this case, one introduces a detuning parameter σ to convert the small-divisor terms into secular terms according to

$$\Omega = 2\omega_0 + \epsilon\sigma \tag{8}$$

Eliminating the terms in equation (7) that produce secular terms in u_1 yields

$$2i\omega_0(D_1A + \mu A) + 3\alpha_2A^2\bar{A} + \frac{1}{2}\alpha_4\bar{A}e^{i\sigma T_1} = 0 \tag{9}$$

Consequently, the solution of equation (7) is

$$\begin{aligned} u_1 = &-\frac{2A\bar{A}}{\omega_0^2}(\alpha_1 + 3\alpha_3A\bar{A}) + \frac{A^2}{3\omega_0^2}(\alpha_1 + 4\alpha_3A\bar{A})e^{2i\omega_0T_0} + \frac{\alpha_2A^3}{8\omega_0^2}e^{3i\omega_0T_0} \\ &+ \frac{\alpha_3A^4}{15\omega_0^2}e^{4i\omega_0T_0} + \frac{\alpha_4A}{2\Omega(\Omega + 2\omega_0)}e^{i(\Omega+\omega_0)T_0} + \text{c.c.} \end{aligned} \tag{10}$$

Substituting the polar form

$$A = \frac{1}{2}ae^{\beta} \tag{11}$$

into equation (9), where a and β are real, and separating real and imaginary parts yields

$$a' + \mu a = -\frac{\alpha_4 a}{4\omega_0} \sin \gamma \tag{12}$$

$$\frac{1}{2} a(\sigma - \gamma') - \frac{3\alpha_2 a^3}{8\omega_0} = \frac{\alpha_4 a}{4\omega_0} \cos \gamma \tag{13}$$

where

$$\gamma = \sigma T_1 - 2\beta \tag{14}$$

Substituting equations (6) and (10) into equation (2) yields the approximate solution

$$\begin{aligned} u = a \cos(\omega_0 t + \beta) + \epsilon \Bigg\{ & \frac{-a^2(4\alpha_1 + 3\alpha_3 a^2)}{8\omega_0^2} + \frac{(\alpha_1 + \alpha_3 a^2) a^2}{6\omega_0^2} \\ & \times \cos(2\omega_0 t + 2\beta) \\ & + \frac{\alpha_2 a^3}{32\omega_0^2} \cos(3\omega_0 t + 3\beta) + \frac{\alpha_3 a^4}{120\omega_0^2} \cos(4\omega_0 t + 4\beta) \\ & + \frac{\alpha_4 a}{2\Omega(\Omega + 2\omega_0)} \cos[(\Omega + \omega_0)t + \beta] \Bigg\} + O(\epsilon^2) \end{aligned} \tag{15}$$

For steady-state solutions, $a' = \gamma' = 0$, and equations (12) and (13) become

$$\mu a = \frac{-\alpha_4 a}{4\omega_0} \sin \gamma \tag{16}$$

$$\frac{1}{2} a\sigma - \frac{3\alpha_2 a^3}{8\omega_0} = \frac{\alpha_4 a}{4\omega_0} \cos \gamma \tag{17}$$

Equations (16) and (17) show that there are two possibilities: $a = 0$ or $a \neq 0$. When $a \neq 0$.

$$\mu = -\frac{\alpha_4}{4\omega_0} \sin \gamma \tag{18}$$

$$\frac{1}{2} \sigma - \frac{3\alpha_2 a^2}{8\omega_0} = \frac{\alpha_4}{4\omega_0} \cos \gamma \tag{19}$$

Squaring equations (18) and (19) and adding the results gives the frequency-response equation

$$\frac{9\alpha_2^2}{64\omega_0^2}a^4 - \frac{3\alpha_2\sigma}{8\omega_0}a^2 + \mu^2 + \frac{\sigma^2}{4} - \frac{\alpha_4^2}{16\omega_0^2} = 0 \tag{20}$$

which, upon solving for a, yields

$$a = [\xi \pm (\xi^2 + \eta)^{1/2}]^{1/2} \tag{21}$$

where

$$\xi = \frac{4\omega_0\sigma}{3\alpha_2}, \qquad \eta = \frac{4}{9\alpha_2^2}(\alpha_4^2 - 16\omega_0^2\mu^2 - 4\omega_0^2\sigma^2) \tag{22}$$

To determine the stability of the trivial solutions, one investigates the solutions of the linearized form of equation (9); that is,

$$2i\omega_0(A' + \mu A) + \frac{1}{2}\alpha_4\bar{A}e^{i\sigma T_1} = 0 \tag{23}$$

Letting

$$A = (B + ib)e^{i\sigma T_1/2} \tag{24}$$

in equation (23), where B and b are real, and separating real and imaginary parts, one obtains

$$B' + \mu B + \Gamma_1 b = 0 \tag{25}$$

$$b' + \mu b + \Gamma_2 B = 0 \tag{26}$$

where

$$\Gamma_1 = -\left(\frac{\sigma}{2} + \frac{\alpha_4}{4\omega_0}\right) \tag{27}$$

$$\Gamma_2 = \frac{\sigma}{2} - \frac{\alpha_4}{4\omega_0} \tag{28}$$

Equations (25) and (26) admit solutions of the form

$$(B, b) = (\hat{B}, \hat{b})e^{\lambda T_1} \tag{29}$$

where $\hat{B}$ and $\hat{b}$ are arbitrary constants and

$$\lambda = -\mu \pm \sqrt{\Gamma_1\Gamma_2} \tag{30}$$

Consequently, a trivial solution is unstable if and only if

$$\Gamma_1\Gamma_2 > \mu^2 \tag{31}$$

and otherwise it is stable.

To determine the stability of the nontrivial solutions, one lets

$$a = a_0 + a_1(T_1), \qquad \gamma = \gamma_0 + \gamma_1(T_1) \tag{32}$$

where a_0 and γ_0 correspond to a nontrivial solution and a_1 and γ_1 are perturbations which are assumed to be small compared with a_0 and γ_0. Substituting equation (32) into equations (12) and (13) and linearizing the resulting equations, one obtains

$$a_1' + \gamma_1\Gamma_1 = 0 \tag{33}$$

$$\gamma_1' + 2\mu\gamma_1 + a_1\Gamma_2 = 0 \tag{34}$$

where

$$\Gamma_1 = a_0\left(\frac{\sigma}{2} - \frac{3\alpha_2 a_0^2}{8\omega_0}\right) \tag{35}$$

$$\Gamma_2 = \frac{3\alpha_2 a_0}{2\omega_0} \tag{36}$$

Consequently, a nontrivial solution is stable if and only if the real parts of both eigenvalues of the coefficient matrix in equations (33) and (34) are less than or equal to zero. Since equations (33) and (34) admit solutions of the form $(a_1, \gamma_1) \propto e^{\lambda T_1}$ provided that

$$\lambda = -\mu \pm \sqrt{\mu^2 + \Gamma_1\Gamma_2} \tag{37}$$

then the steady-state solutions are unstable if and only if

$$\Gamma_1\Gamma_2 > 0 \tag{38}$$

and otherwise they are stable.

It follows from equation (15) that, when $\epsilon \to 0$, then

$$u \to a\cos(\tfrac{1}{2}\Omega t - \tfrac{1}{2}\gamma) \tag{39}$$

where a and γ are given by

$$a = [\xi \pm (\xi^2 + \eta)^{1/2}]^{1/2} \tag{40}$$

$$\gamma = \tan^{-1}\left(\frac{8\omega_0\mu}{3\alpha_2 a^2 - 4\omega_0\sigma}\right) \tag{41}$$

Noting that equation (39) is the solution of equation (1) in the case of the steady state. Also, as $t \to \infty$, the solution is bounded.

4. THE GENERALIZED SYNCHRONIZATION METHOD

For this method see Balbi (1973) and Elnaggar and Hamd-Allah (1982). When $\epsilon = 0$, the solution of equation (1) can be written as

$$u = a\cos(\omega_0 t + \varphi) \tag{42}$$

where a and φ are constants. It follows from equation (42) that

$$\dot{u} = -\omega_0 a \sin(\omega_0 t + \varphi) \tag{43}$$

When $\epsilon \neq 0$, we assume that the solution of equation (1) is still given by equation (42), but with time-varying a and φ. Differentiating equation (42) with respect to t and recalling that a and φ are functions of t, we have

$$\dot{u} = -\omega_0 a \sin(\omega_0 t + \varphi) + \dot{a}\cos(\omega_0 t + \varphi) - a\dot{\varphi}\sin(\omega_0 t + \varphi) \tag{44}$$

Comparing equation (44) with equation (43), we conclude that

$$\dot{a}\cos(\omega_0 t + \varphi) - a\dot{\varphi}\sin(\omega_0 t + \varphi) = 0 \tag{45}$$

Differentiating equation (43) with respect to t, we obtain

$$\ddot{u} = -\omega_0^2 a\cos(\omega_0 t + \varphi) - \omega_0\dot{a}\sin(\omega_0 t + \varphi) - \omega_0 a\dot{\varphi}\cos(\omega_0 t + \varphi) \tag{46}$$

Substituting for u, $\dot{u}$, and $\ddot{u}$ from equations (42), (43), and (46) into equation (1), we have

$$\begin{aligned}
&\dot{a}\sin(\omega_0 t + \varphi) + a\dot{\varphi}\cos(\omega_0 t + \varphi) \\
&\quad = \epsilon\left[-2\mu a\sin(\omega_0 t + \varphi) + \frac{\alpha_1 a^2}{\omega_0}\cos^2(\omega_0 t + \varphi)\right. \\
&\qquad + \frac{\alpha_2 a^3}{\omega_0}\cos^3(\omega_0 t + \varphi) + \frac{\alpha_3 a^4}{\omega_0}\cos^4(\omega_0 t + \varphi) \\
&\qquad \left. + \frac{\alpha_4 a}{\omega_0}\cos(\omega_0 t + \varphi)\cos\Omega t\right]
\end{aligned} \tag{47}$$

Solving equations (45) and (47) for $\dot{a}$ and $\dot{\varphi}$ and using the trigonometric identities gives the following variational equations:

$$\begin{aligned}
\dot{a} = \epsilon&\left[-\mu a + \frac{2\alpha_1 a^2 + \alpha_3 a^4}{8\omega_0}\sin\psi_0 + \frac{\alpha_2 a^3}{4\omega_0}\sin 2\psi_0 + \mu a\cos 2\psi_0\right. \\
&+ \frac{4\alpha_1 a^2 + 3\alpha_3 a^4}{16\omega_0}\sin 3\psi_0 + \frac{\alpha_2 a^3}{8\omega_0}\sin 4\psi_0 + \frac{\alpha_3 a^4}{16\omega_0}\sin 5\psi_0 \\
&\left. + \frac{\alpha_4 a}{4\omega_0}\sin(\psi + 2\psi_0) - \frac{\alpha_4 a}{4\omega_0}\sin(\psi - 2\psi_0)\right]
\end{aligned} \tag{48}$$

$$\dot{\varphi} = \epsilon\left[\frac{3\alpha_2 a^2}{8\omega_0} + \frac{6\alpha_1 a + 5\alpha_3 a^3}{8\omega_0}\cos\psi_0 - \mu\sin 2\psi_0 + \frac{\alpha_2 a^2}{2\omega_0}\cos 2\psi_0 \right.$$
$$+ \frac{4\alpha_1 a + 5\alpha_3 a^3}{16\omega_0}\cos 3\psi_0 + \frac{\alpha_2 a^2}{8\omega_0}\cos 4\psi_0 + \frac{\alpha_3 a^3}{16\omega_0}\cos 5\psi_0$$
$$\left. + \frac{\alpha_4}{4\omega_0}\cos(\psi + 2\psi_0) + \frac{\alpha_4}{4\omega_0}\cos(\psi - 2\psi_0) + \frac{\alpha_4}{2\omega_0}\cos\psi\right] \tag{49}$$

where

$$\psi_0 = \omega_0 t + \varphi \tag{50}$$

$$\psi = \Omega t \tag{51}$$

For principal parametric synchronization (i.e., $\Omega \approx 2\omega_0$), then from equations (48) and (49) we retain only the constant terms and the terms of small frequency; thus we have

$$\bar{F}_1(y, t) = \bar{f}(y, t) = \begin{bmatrix} \bar{f}_{1a}(y, t) \\ \bar{f}_{1\varphi}(y, t) \end{bmatrix}, \qquad y = \begin{bmatrix} A \\ \Phi \end{bmatrix}$$

where

$$\bar{f}_{1a}(y, t) = -\mu A - \frac{\alpha_4 A}{4\omega_0}\sin[(\Omega - 2\omega_0)t - 2\Phi] \tag{52}$$

$$\bar{f}_{1\varphi}(y, t) = \frac{3\alpha_2 A^2}{8\omega_0} + \frac{\alpha_4}{4\omega_0}\cos[(\Omega - 2\omega_0)t - 2\Phi] \tag{53}$$

and the terms of higher frequency are

$$\tilde{f}(y, t) = \begin{bmatrix} \tilde{f}_{1a}(y, t) \\ \tilde{f}_{1\varphi}(y, t) \end{bmatrix}$$

where

$$\tilde{f}_{1a}(y, t) = \frac{2\alpha_1 A^2 + \alpha_3 A^4}{8\omega_0}\sin\psi_0 + \frac{\alpha_2 A^3}{4\omega_0}\sin 2\psi_0 + \mu A\cos 2\psi_0$$
$$+ \frac{4\alpha_1 A^2 + 3\alpha_3 A^4}{16\omega_0}\sin 3\psi_0 + \frac{\alpha_2 A^3}{8\omega_0}\sin 4\psi_0 + \frac{\alpha_3 A^4}{16\omega_0}\sin 5\psi_0$$
$$+ \frac{\alpha_1 A}{4\omega_0}\sin(\psi + 2\psi_0) \tag{54}$$

$$\tilde{f}_{1\varphi}(y,t) = \frac{6\alpha_1 A + 5\alpha_3 A^3}{8\omega_0}\cos\psi_0 - \mu\sin 2\psi_0 + \frac{\alpha_2 A^2}{2\omega_0}\cos 2\psi_0$$
$$+ \frac{4\alpha_1 A + 5\alpha_3 A^3}{16\omega_0}\cos 3\psi_0 + \frac{\alpha_2 A^2}{8\omega_0}\cos 4\psi_0$$
$$+ \frac{\alpha_3 A^3}{16\omega_0}\cos 5\psi_0 + \frac{\alpha_4}{4\omega_0}\cos(\psi + 2\psi_0) + \frac{\alpha_4}{2\omega_0}\cos\psi \qquad (55)$$

Then the reduced system to the first approximation takes the form

$$\begin{bmatrix} \dot{A} \\ \Phi \end{bmatrix} = \epsilon\bar{F}_1(y,t)$$

i.e.,

$$\dot{A} = -\epsilon\left\{\mu A + \frac{\alpha_4 A}{4\omega_0}\sin[(\Omega - 2\omega_0)t - 2\Phi]\right\} \qquad (56)$$

$$\dot{\Phi} = \epsilon\left\{\frac{3\alpha_2 A^2}{8\omega_0} + \frac{\alpha_4}{4\omega_0}\cos[(\Omega - 2\omega_0)t - 2\Phi]\right\} \qquad (57)$$

Since t appears explicitly in equations (56) and (57), they are called a nonautonomous system. It is convenient to eliminate the explicit dependence on t, thereby transforming there equations into an autonomous system. This can be accomplished by introducing the new dependent variable γ defined by

$$\gamma = (\Omega - 2\omega_0)t - 2\Phi \qquad (58)$$

Substituting equations (8) and (58) into equations (56) and (57), one obtains the autonomous system that describes the modulation of the amplitude and the phase:

$$\dot{A} = -\epsilon\mu A - \frac{\alpha_4\epsilon A}{4\omega_0}\sin\gamma \qquad (59)$$

$$\frac{1}{2}(\epsilon\sigma - \dot{\gamma}) = \frac{3\alpha_2\epsilon A^2}{8\omega_0} + \frac{\alpha_4\epsilon}{4\omega_0}\cos\gamma \qquad (60)$$

For steady-state solutions, we put $\dot{A} = \dot{\gamma} = 0$; then equations (59) and (60) become, when $A \neq 0$,

$$\mu = \frac{-\alpha_4}{4\omega_0}\sin\gamma \qquad (61)$$

$$\frac{1}{2}\sigma - \frac{3\alpha_2 A^2}{8\omega_0} = \frac{\alpha_4}{4\omega_0}\cos\gamma \qquad (62)$$

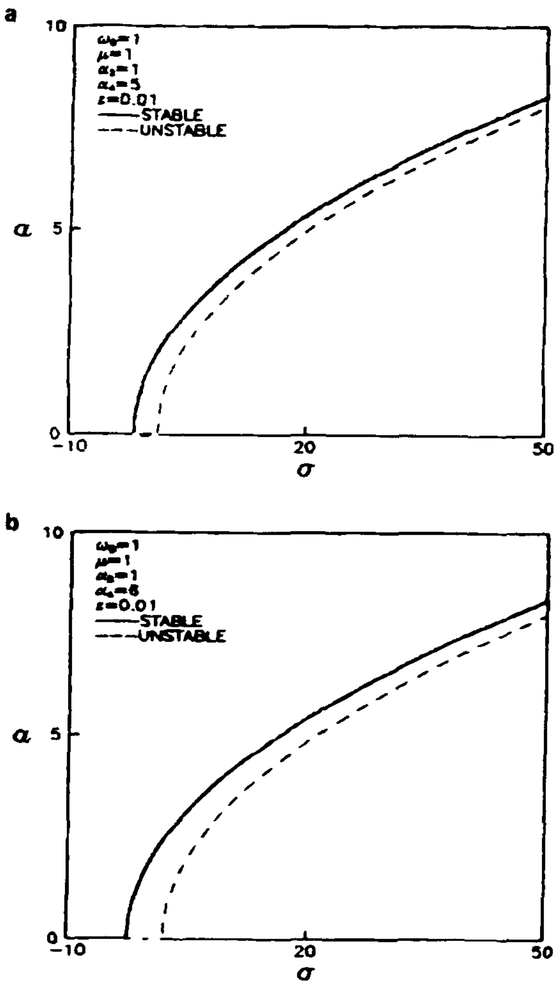


Fig. 1. Frequency response curves for subharmonic oscillations of order 1/2.

Squaring equations (61) and (62) and adding the results gives the frequency-response equation:

$$\frac{9\alpha_2^2}{64\omega_0^2}A^4 - \frac{3\alpha_2\sigma}{8\omega_0}A^2 + \mu^2 + \frac{\sigma^2}{4} - \frac{\alpha_4^2}{16\omega_0^2} = 0 \tag{63}$$

which is in full agreement with equation (20) obtained by using the method of multiple scales.

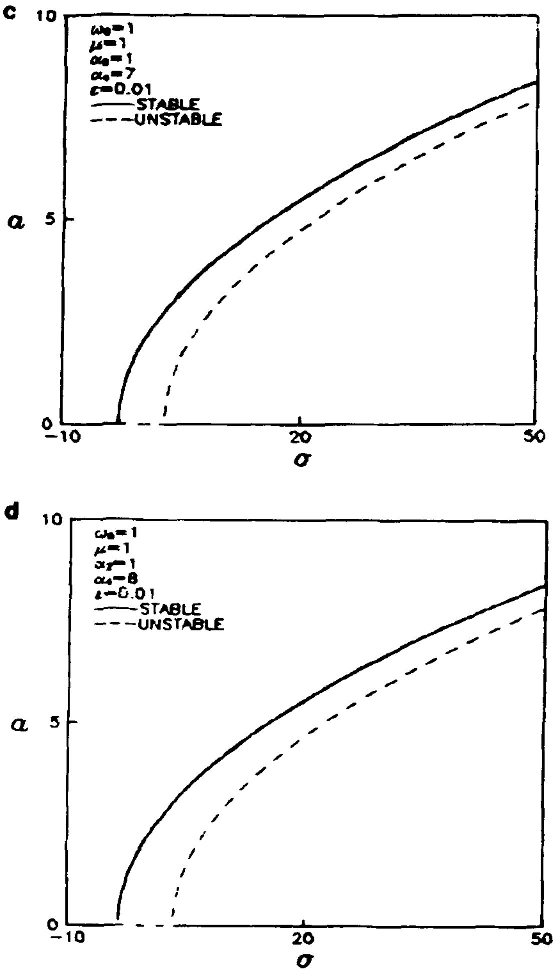


Fig. 1. Continued.

Following a procedure similar to that in the preceding section, one obtains the following eigenvalues that determine the stability of the steady-state solutions:

$$\lambda = -\mu \pm \sqrt{\mu^2 + \Gamma_1\Gamma_2} \tag{64}$$

where

$$\Gamma_1 = \epsilon a_0 \left[\frac{1}{2}\sigma - \frac{3\alpha_2 a_0^2}{8\omega_0}\right] \tag{65}$$

$$\Gamma_2 = \frac{3\alpha_2 \epsilon a_0}{2\omega_0} \tag{66}$$

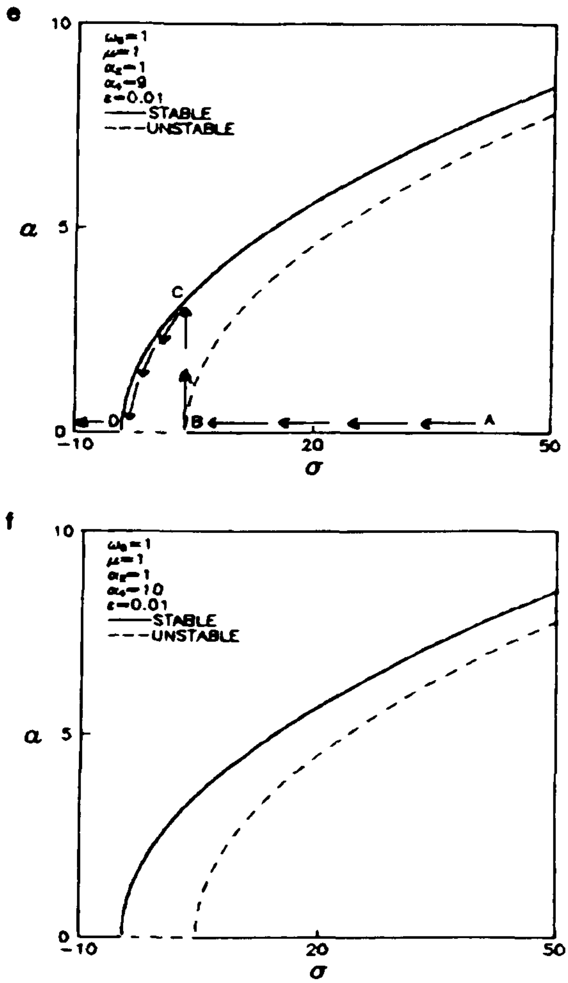


Fig. 1. Continued.

which are in excellent agreement with equations (35) and (36) obtained by using the method of multiple scales. Note that in the case of the multiple scales we have $T_1 = \epsilon t$ and $T_0 = t$.

In order to establish the approximate amplitude and the approximate phase, we find the function $\tilde{G}_1(y, t)$ as follows:

$$\tilde{G}_1(y, t) = \begin{bmatrix} \tilde{G}_{1a}(y, t) \\ \tilde{G}_{1\varphi}(y, t) \end{bmatrix}$$

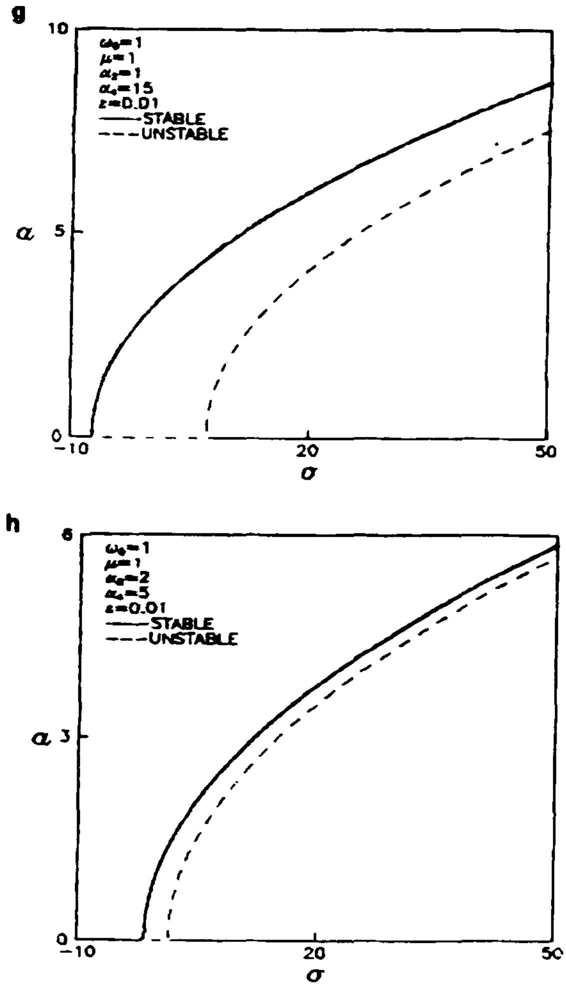


Fig. 1. Continued.

where

$$
\begin{aligned}
\tilde{G}_{1a}(y, t) &= \int \tilde{f}_{1a}(y, t)\, dt \\
&= -\left\{ \frac{2\alpha_1 A^2 + \alpha_3 A^4}{8\omega_0^2} \cos \psi_0 + \frac{\alpha_2 A^3}{8\omega_0^2} \cos 2\psi_0 - \frac{\mu A}{2\omega_0} \sin 2\psi_0 \right. \\
&\quad + \frac{4\alpha_1 A^2 + 3\alpha_3 A^4}{48\omega_0^2} \cos 3\psi_0 + \frac{\alpha_2 A^3}{32\omega_0^2} \cos 4\psi_0 + \frac{\alpha_3 A^4}{8\omega_0^2} \cos 5\psi_0 \\
&\quad \left. + \frac{\alpha_4 A}{4\omega_0(\Omega + 2\omega_0)} \cos(\psi + 2\psi_0) \right\}
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\tilde{G}_{1\varphi}(y, t) &= \int \tilde{f}_{1\varphi}(y, t)\, dt \\
&= \frac{6\alpha_1 A + 5\alpha_3 A^3}{8\omega_0^2} \sin \psi_0 + \frac{\mu}{2\omega_0} \cos 2\psi_0 + \frac{\alpha_2 A^2}{4\omega_0} \sin 2\psi_0 \\
&\quad + \frac{4\alpha_1 A + 5\alpha_3 A^3}{48\omega_0^2} \sin 3\psi_0 + \frac{\alpha_2 A^2}{32\omega_0^2} \sin 4\psi_0 + \frac{\alpha_3 A^3}{80\omega_0^2} \sin 5\psi_0 \\
&\quad + \frac{\alpha_4}{4\omega_0(\Omega + 2\omega_0)} \sin(\psi + 2\psi_0) + \frac{\alpha_4}{2\omega_0 \Omega} \sin \psi
\end{aligned}
\tag{68}
$$

Then the amplitude and the phase to the first approximation are defined by

$$
\begin{aligned}
a(t) &= A + \epsilon \tilde{G}_{1a}(y, t) \\
&= A - \epsilon \left\{ \frac{2\alpha_1 A^2 + \alpha_3 A^4}{8\omega_0^2} \cos \psi_0 + \frac{\alpha_2 A^3}{8\omega_0^2} \cos 2\psi_0 - \frac{\mu A}{2\omega_0} \sin 2\psi_0 \right. \\
&\quad + \frac{4\alpha_1 A^2 + 3\alpha_3 A^4}{48\omega_0^2} \cos 3\psi_0 + \frac{\alpha_2 A^3}{32\omega_0^2} \cos 4\psi_0 + \frac{\alpha_3 A^4}{80\omega_0^2} \cos 5\psi_0 \\
&\quad \left. + \frac{\alpha_4 A}{4\omega_0(\Omega + 2\omega_0)} \cos(\psi + 2\psi_0) \right\}
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
\varphi(t) &= \Phi + \epsilon \tilde{G}_{1\varphi}(y, t) \\
&= \Phi + \epsilon \left\{ \frac{6\alpha_1 A + 5\alpha_3 A^3}{8\omega_0^2} \sin \psi_0 + \frac{\mu}{2\omega_0} \cos 2\psi_0 + \frac{\alpha_2 A^2}{4\omega_0} \sin 2\psi_0 \right. \\
&\quad + \frac{4\alpha_1 A + 5\alpha_3 A^3}{48\omega_0^2} \sin 3\psi_0 + \frac{\alpha_2 A^2}{32\omega_0^2} \sin 4\psi_0 + \frac{\alpha_3 A^3}{80\omega_0^2} \sin 5\omega_0 \\
&\quad \left. + \frac{\alpha_4}{4\omega_0(\Omega + 2\omega_0)} \sin(\psi + 2\psi_0) + \frac{\alpha_4}{2\omega_0 \Omega} \sin \psi \right\}
\end{aligned}
\tag{70}
$$

Then the approximate solution is

$$
\begin{aligned}
u(t) &= a(t) \cos[\omega_0 t + \varphi(t)] \\
&= [A + \epsilon \tilde{G}_{1a}(y, t)] \cos[\omega_0 t + \Phi + \epsilon \tilde{G}_{1\varphi}(y, t)]
\end{aligned}
$$

i.e.,

$$u(t) = A\cos\psi_0 + \epsilon\left\{-\frac{4\alpha_1 A^2 + 3\alpha_3 A^4}{8\omega_0^2} + \frac{\mu A}{2\omega_0}\sin\psi_0 - \frac{3\alpha_2 A^3}{16\omega_0^2}\cos\psi_0\right.$$
$$+ \frac{\alpha_1 A^2 + \alpha_3 A^4}{6\omega_0^2}\cos 2\psi_0 + \frac{\alpha_2 A^3}{32\omega_0^2}\cos 3\psi_0 + \frac{\alpha_4 A^4}{120\omega_0^2}\cos 4\psi_0$$
$$\left. + \frac{\alpha_4 A}{2\Omega(\Omega + 2\omega_0)}\cos(\psi + \psi_0) - \frac{\alpha_4 A}{4\omega_0\Omega}\cos(\psi - \psi_0)\right\} + O(\epsilon^2) \quad (71)$$

which is in excellent agreement with the solution obtained by using the method of multiple scales and defined by equation (15).

When $\epsilon \to 0$, then $\Omega \to 2\omega_0$, $a(t) \to A$, $\varphi(t) \to \Phi$, and

$$u(t) \to A\cos\left(\frac{1}{2}\Omega t + \Phi\right) \quad (72)$$

where

$$A = [\xi \pm (\xi^2 + \eta)^{1/2}]^{1/2}, \quad (73)$$

$$\Phi = -\frac{1}{2}\gamma \quad (74)$$

$$\xi = \frac{4\omega_0\sigma}{3\alpha_2} \quad (75)$$

$$\eta = \frac{4}{9\alpha_2^2}(\alpha_4^2 - 16\omega_0^2\mu^2 - 4\omega_0^2\sigma^2) \quad (76)$$

$$\gamma = \tan^{-1}\left(\frac{8\omega_0\mu}{3\alpha_2 A^2 - 4\omega_0\sigma}\right) \quad (77)$$

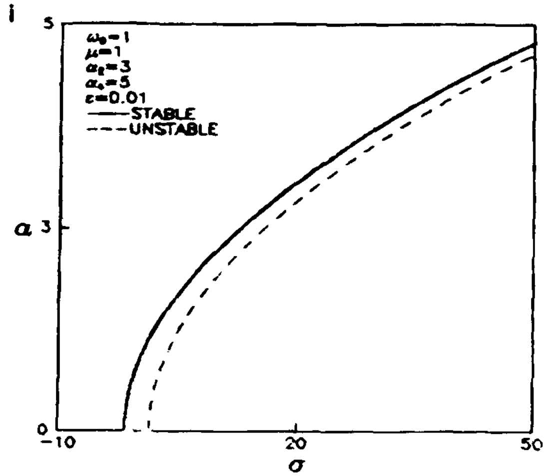
These equations are in full agreement with the corresponding equations obtained by using the method of multiple scales in the preceding section.

5. NUMERICAL RESULTS AND DISCUSSION

The frequency response equation (20) [which is in full agreement with equation (63)] is a nonlinear algebraic equation in the amplitude a. This equation is solved numerically by using the bisection method (Gerald, 1980).

The numerical results are represented by Figures 1A–1P, which represent the variation of the amplitude a with the detuning parameter σ for given values of the other parameters. In all figures, the solid lines represent stable solutions, while the dashed lines represent unstable solutions.

From the geometry of the figures we observe that the frequency-response curves consist of two branches; the left one is stable and the right one is unstable. These curves are bent to the right; the bending leads to multivalued solutions and hence to a jump phenomenon. Also, there exist stable and unstable trivial solutions. As the coefficient of parametric α_4 excitation



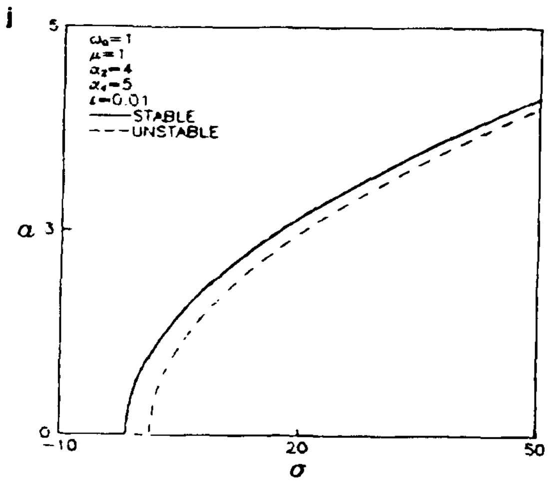


Fig. 1. Continued.

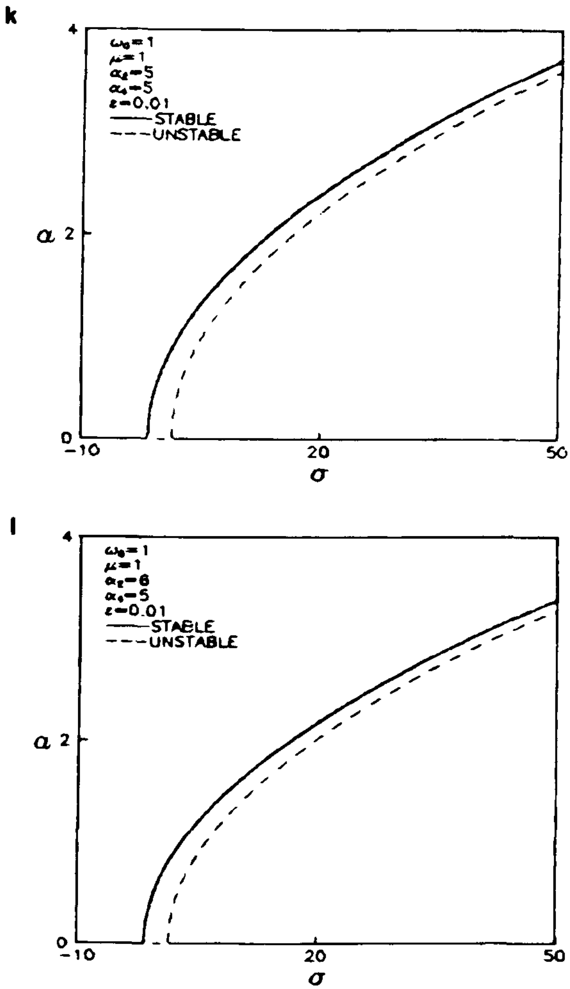


Fig. 1. Continued.

increases while the other parameters remain constant, the branches of the response curve diverge from each other, and the region of unstable trivial solutions increases (Figs. 1A–1G). As the coefficient of the cubic term α_2 increases while the other parameters remain constant, the solutions have a small magnitude (Figs. 1H–1M). As the coefficient of the damping term μ decreases while the other parameters remain constant, the response curves are not strongly affected and they shift slowly to the left, as shown in Figs. 1N–1P.

Finally, we explain the jump phenomenon: for example, in Fig. 1E, as σ is reduced from a value corresponding to the point A, the amplitude remains

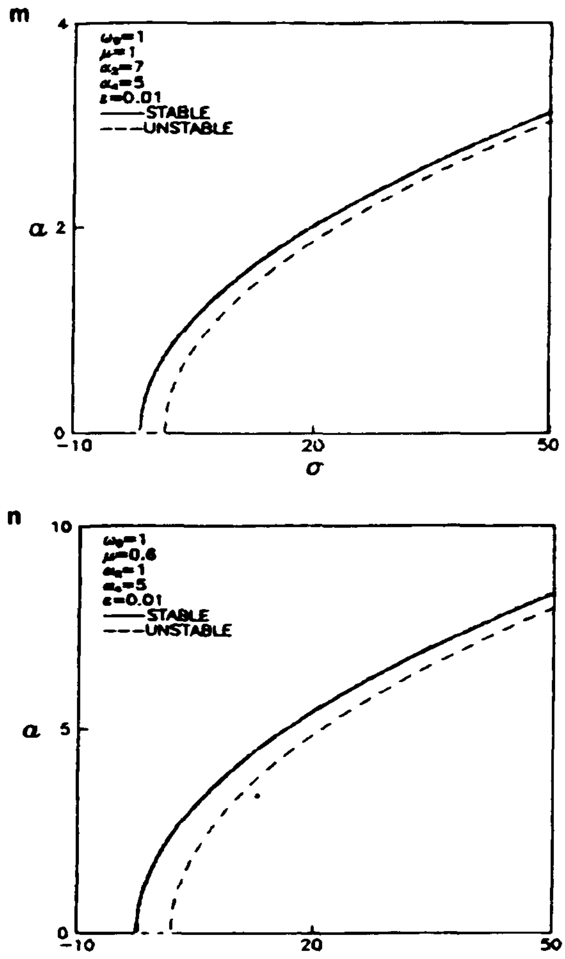


Fig. 1. Continued.

zero until the point B is reached. As σ is decreased further, a jump takes place from the point B to the point C. Then, as σ decreases further, the amplitude decreases slowly. Also, we note that the region between the points B and D represents unstable trivial solutions, while for the other regions we have stable trivial solutions.

6. SUMMARY AND CONCLUSION

Two approximate methods (multiple scales and generalized synchronization) have been used to obtain a uniform first-order (two-term) expansion

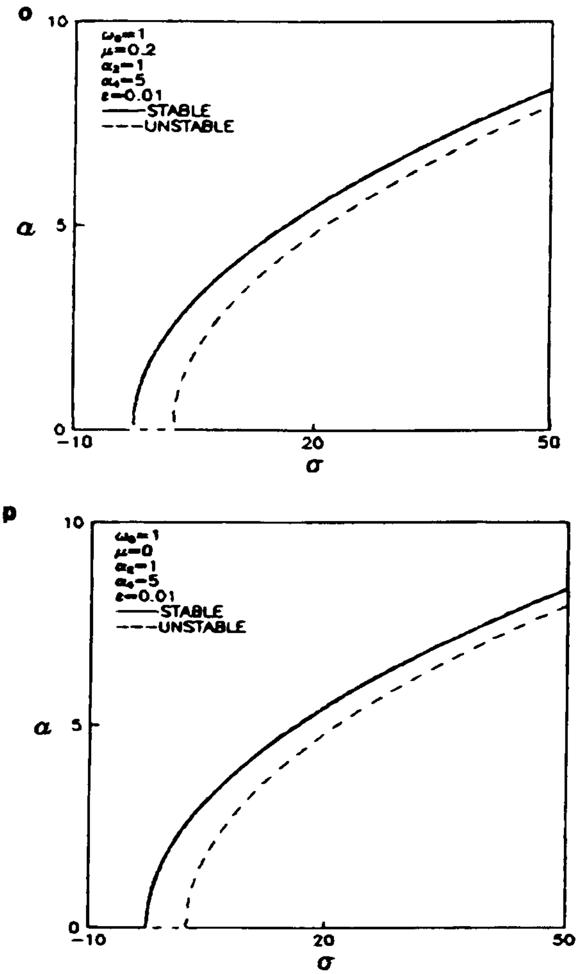


Fig. 1. Continued.

for a one-degree-of-freedom system with quadratic, cubic, and quartic nonlinearities under a parametric excitation. Two first-order ordinary differential equations which describe the modulation of the amplitudes and the phases were derived. Steady-state solutions (periodic solutions) and their stability were obtained. Numerical solutions were found by using the bisection method, and are plotted in Figs. 1A–1P. The results obtained by the two methods are in excellent agreement. The following conclusions can be deduced from the analysis:

1. The frequency-response curves consist of two branches; the left one is stable and the right one is unstable. These curves are bent to the right; the bending leads to multivalued solutions and hence to a jump phenomenon.

2. As the coefficient of parametric excitation α_4 increases while the other parameters remain constant, the branches of the response curve diverge from each other, and the region of unstable trivial solutions increases.

3. As the coefficient of cubic term α_2 increases while the other parameters remain constant, the solutions have a small magnitude.

4. As the coefficient of damping term μ decreases while the other parameters remain constant, the response curves are not strongly affected, and shift slowly to the left.

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